Relation between observability and differential embeddings for nonlinear dynamics

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In the analysis of a scalar time series, which lies on an m-dimensional object, a great number of techniques will start by embedding such a time series in a d-dimensional space, with $d > m$. Therefore there is a coordinate transformation Φ_s from the original phase space to the embedded one. The embedding space depends on the observable $s(t)$. In theory, the main results reached are valid regardless of $s(t)$. In a number of practical situations, however, the choice of the observable does influence our ability to extract dynamical information from the embedded attractor. This may arise in problems in nonlinear dynamics such as model building, control and synchronization. To some degree, ease of success will depend on the choice of the observable simply because it is related to the observability of the dynamics. In this paper the observability matrix for nonlinear systems, which uses Lie derivatives, is revisited. It is shown that such a matrix can be interpreted as the Jacobian matrix of Φ_s—the map between the original phase space and the differential embedding induced by the observable—thus establishing a link between observability and embedding theory.

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I. INTRODUCTION

When an experimental mth-order dynamical system is investigated, the m physical quantities (state variables) should be all measured, at least in principle, to have a complete description of the state of the system under study. Unfortunately, in most experimental situations, only a single physical quantity is measured. The first step in the analysis is therefore to reconstruct a phase space from this scalar time series by using some type of coordinate system. Commonly used coordinates use time delays or time derivatives [1]. If the dimension of the embedding space is sufficiently large, the reconstructed trajectory is expected to have the same properties as the trajectory embedded in the original phase space [2].

A great number of techniques developed for studying nonlinear dynamical systems start with such an embedding and the main results reached are valid, in general, regardless of the observable chosen [2]. In a number of practical situations, however, the choice of the observable does influence our ability to extract dynamical information from the embedded attractor. This dependence results from the complexity of the coupling between the dynamical variables, and the symmetry properties the original system may have [3]. The coupling complexity can be estimated with observability indices [4,5] and, consequently, the variables can be ranked. The dependence on the choice of the observable has direct bearing on problems in nonlinear dynamics such as model building [6,7], synchronization [8] and control [9,10]. In all these cases, ease of success and overall performance will depend to some degree on the choice of observable which is related to the observability of the dynamics. Despite the potential practical importance of this issue, observability is not a problem commonly addressed in the community of nonlinear dynamics. It is conjectured that one of the reasons for this is probably that, although concepts such as embedding are quite familiar to most researchers in nonlinear dynamics, this is not necessarily true concerning observability theory.

In previous works the issue of observability of nonlinear dynamical systems [3,5] was addressed simply by locally applying the theory of linear systems [4] and taking time averages along an orbit in state space. In [3] it was shown that ergodicity applies, that is, the results remain qualitatively unchanged if averaging is taken over an ensemble of orbits rather than over time. This paper extends the previous results by using a different definition for the observability matrix, defined using Lie derivatives [11], which is better suited to analyze nonlinear systems. Such an extension has two main consequences. First, in many cases it emphasizes the effects of nonlinearity on the computed observability indices. Second, and most important, it provides a link with embedding theory, which seems to be a useful interpretation that has not been reported so far in the literature.

This paper is organized as follows. Section II A gives some basic definitions of observability and Sec. II B provides a link between observability and embedding theory. In Sec. III an example is discussed and the observability matrix is interpreted as the Jacobian matrix of the coordinate transformation between the original phase space and the differential embedding induced by the given variable. Section III A brings to light the link between the observability indices and the ability to build a good global model from a single time series. Section IV is devoted to five different examples. Finally, Sec. V provides the main conclusions of the paper.

II. OBSERVABILITY FOR NONLINEAR SYSTEMS

A. Concepts

Briefly, a system is observable if the full state can be found based on the system output [27], which could be one of the state variables or a function of the state vector s

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$=h(\boldsymbol{x})$. This concept is of great importance in a number of applications. If a certain system is poorly observable from a given variable, the use of such a variable hampers building a useful model or analyzing the system from data [12,13]. Similarly, the use of Kalman filtering schemes to estimate unobserved states requires the choice of a variable to drive the filter [14]. The performance of the filter will usually depend on how observable is the system from the chosen variable.

For linear systems, observability conditions are generally easy to compute [4]. Although the concepts of controllability and observability for linear systems were developed in the early 1960s, it was not until the 1970s that the nonlinear counterparts started to be developed for which Lie algebra became an important tool [15–17]. Now consider a nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),$$

$$s(t) = h(\boldsymbol{x}), \tag{1}$$

with $\boldsymbol{f}:\mathbb{R}^m \to \mathbb{R}^m$. Differentiating $s(t)$ yields

$$\dot{s}(t) = \frac{d}{dt}h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} = \frac{\partial h}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}) = \mathcal{L}_f h(\boldsymbol{x}). \tag{2}$$

$\mathcal{L}_f h(\boldsymbol{x})$ is the Lie derivative of h along the vector field $\boldsymbol{f}$. The jth order Lie derivative is given by

$$\mathcal{L}_f^j h(\boldsymbol{x}) = \frac{\partial \mathcal{L}_f^{j-1} h(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{f}(\boldsymbol{x}), \tag{3}$$

where $\mathcal{L}_f^0 h(\boldsymbol{x}) = h(\boldsymbol{x})$. The time derivatives of s can be written in terms of Lie derivatives as $s^{(j)} = \mathcal{L}_f^j h(\boldsymbol{x})$. The observability matrix can be written as

$$O_s(\boldsymbol{x}) = \begin{bmatrix} \dfrac{\partial \mathcal{L}_f^0 h(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \vdots \\ \dfrac{\partial \mathcal{L}_f^{m-1} h(\boldsymbol{x})}{\partial \boldsymbol{x}} \end{bmatrix}, \tag{4}$$

where the index s has been used to emphasize that $O_s(\boldsymbol{x})$ refers to the system observed from $s(t)$.

System (1) is said to be *observable* if every pair of different initial conditions $\boldsymbol{x}_{0_1}$ and $\boldsymbol{x}_{0_2}$ is distinguishable with respect to the measured time series $s(t), t \geqslant 0$. That is, if the system is observable, it is possible to trace back every single initial condition given only the measured time series $s(t), t \geqslant 0$, or still, $h(\boldsymbol{x}_{0_1})|_{t\geqslant 0} \neq h(\boldsymbol{x}_{0_2})|_{t\geqslant 0}$ iff $\boldsymbol{x}_{0_1} \neq \boldsymbol{x}_{0_2}$.

In order to check the observability of a nonlinear system, it is thus convenient to investigate the map $\Phi_s:\mathbb{R}^m(\boldsymbol{x}) \mapsto \mathbb{R}^d(s(t), s^{(1)}, \dots, s^{(d-1)})$. If Φ_s is invertible (injective), for a given embedding dimension d, it is possible to reconstruct the state from $s(t)$. Although it is often impossible to check the global invertibility for general nonlinear maps, the Rank theorem provides a sufficient condition for local invertibility. The map Φ_s is locally invertible at $\boldsymbol{x}_0$ if the Jacobian has full rank, that is, if

$$\operatorname{rank}\left(\left.\frac{\partial \Phi_s}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_0} \right) = m. \tag{5}$$

Hence, the system is locally observable if condition (5) holds.

B. Link between observability and embedding

The main point of this paper is to point out that the matrix in (5) used to test for the local invertibility of the embedding is, in fact, the observability matrix that corresponds to $s(t) = h(\boldsymbol{x})$. In other words the Jacobian of the map between the original phase space $\mathbb{R}^m(\boldsymbol{x})$ and the reconstructed phase space (also called the differential embedding) $\mathbb{R}^d(\boldsymbol{X})$ where $\boldsymbol{X} = (s(t), s^{(1)}, \dots, s^{(d-1)})$, is the *observability matrix* O_s defined for nonlinear systems [18]. The observability of a dynamical system from an observable $s(t)$ is therefore directly related to the existence of singularities in Φ_s.

It is important to realize that this connection is not arrived at if the former definition of the observability matrix is used, that is, the definition put forward in [3,5]. Therefore it seems that the definition (4) has two benefits. First, it is more adequate for nonlinear systems than to locally apply the linear definition and subsequently average along a trajectory. Second, it provides a clear link, which seems to be lacking in the literature, between observability and embedding theories for the case of continuous-time systems.

The standard definitions of observability and controllability are "yes" or "no" measures, that is, the system is either observable or not [4]. In practice, however, a system may gradually become unobservable as a parameter is varied or, for nonlinear systems, there are regions in phase space that are less observable than others. Following our previous work, we quantify the degree of observability with the observability index, defined as [3,4]

$$\delta_s(\boldsymbol{x}) = \frac{|\lambda_{\min}[O_s^T O_s, \boldsymbol{x}(t)]|}{|\lambda_{\max}[O_s^T O_s, \boldsymbol{x}(t)]|}, \tag{6}$$

where $\lambda_{\max}[O_s^T O_s, \boldsymbol{x}(t)]$ indicates the maximum eigenvalue of matrix $O_s^T O_s$ estimated at point $\boldsymbol{x}(t)$ (likewise for $\lambda_{\min}$). Then $0 \leqslant \delta(\boldsymbol{x}) \leqslant 1$, and the lower bound is reached when the system is unobservable at point $\boldsymbol{x}$. It should be noticed that index (6) is a type of condition number of the matrix $O_s^T O_s$, which has been called *distortion matrix* in [19].

It will be convenient to summarize the observability attained from a given variable using a value averaged along an orbit. In this respect, the following definition is considered:

$$\delta_s = \frac{1}{T}\sum_{t=0}^{T} \delta_s(\boldsymbol{x}(t)), \tag{7}$$

where T is the final time considered and, without loss of generality the initial time was set to be $t=0$. In what follows observability indices δ will be calculated for several systems with diverse dynamical properties. The reader should bear in mind, however, that the observability indices are local quantities and that taking the average is useful inasmuch as it portrays an overall picture. Another important remark is that the observability indices do not have absolute interpretation but rather only a relative value. Thus values of δ_s [for different choices of $s(t)$] should be compared within the context of a single system.

In our previous work on observability, these indices were computed using the powers of the system Jacobian matrix [3]. Here the matrix $O_s(\boldsymbol{x})$ is evaluated from (4) along a trajectory $\boldsymbol{x}$ on an attractor and, using (6) and (7), the observability indices will be computed.

III. A SIMPLE EXAMPLE WITH A NONLINEAR SYSTEM

In this section it is desired to apply the main ideas described in Sec. II and illustrate how to affect the modeling of nonlinear dynamics from a single measurand. To this end, let us start with the Rössler system [20] :

$$\dot{x} = -y - z,$$

$$\dot{y} = x + ay,$$

$$\dot{z} = b + z(x - c), \tag{8}$$

where (a,b,c) are the bifurcation parameters. In what follows, each dynamical variable of (8) will successively be considered as the observable. In each case a coordinate transformation map Φ_s will be used in the analysis.

When the system is observed by means of a measurement function such as $s(t)=h(x,y,z)=y(t)$, the corresponding observability matrix is, see (4),

$$O_y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ a & a^2-1 & -1 \end{bmatrix}. \tag{9}$$

Note that O_y is a constant matrix which does not depend on the dynamical variables. Moreover, this matrix is nonsingular and consequently the Rössler system is therefore observable from the y variable at any point of the phase space.

Now let us analyze the system from a differential embedding point of view. The respective coordinate transformation $\Phi_y:\mathbb{R}^3(x,y,z)\mapsto\mathbb{R}^3_y(X,Y,Z)$ is

$$\Phi_y = \left| \begin{array}{l} X = y, \\ Y = \dot{y} = x + ay, \\ Z = \ddot{y} = ax + (a^2-1)y - z, \end{array} \right. \tag{10}$$

and it is very easy to check that O_y in (9) is the Jacobian matrix of Φ_y in (10), that is, $\mathcal{J}(\Phi_y)=O_y$. Map Φ_y defines a *global* diffeomorphism since it is injective and its Jacobian matrix has a determinant which never vanishes. Consequently, when there is a diffeomorphism between the original phase space and the differential embedding induced by an observable, the system can be observed from such a variable at any point in space. From an observability point of view, this is the best situation ever.

When the Rössler system is observed from the x variable, the observability matrix is

$$O_x(\boldsymbol{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ -1-z & -a & c-x \end{bmatrix}. \tag{11}$$

We can easily check that $O_x(\boldsymbol{i})=\mathcal{J}(\Phi_x)$. The determinant $\det(O_x(\boldsymbol{x}))$ vanishes at $x=a+c$ [28]. Φ_x thus defines only a *local* diffeomorphism. The Rössler system is therefore not observable from the x variable over all the phase space.

In a similar way, when $C_z=[0\ 0\ 1]$, the corresponding observability matrix becomes

$$O_z(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 & 1 \\ z & 0 & x-c \\ b+2z(x-c) & -z & (x-c)^2-y-2z \end{bmatrix}. \tag{12}$$

This matrix is not constant and $\det(O_z)=-z^2$, which obviously vanishes for $z=0$. The Rössler system is therefore not everywhere observable from the z-variable. In addition, the plane $z=0$ is an order-two singular set. Consequently, the observability matrix $O_z(\boldsymbol{x})$ is approximately rank deficient even close to the singular plane $z=0$. As expected, we have $\mathcal{J}(\Phi_z)=O_z(\boldsymbol{x})$.

For the three dynamical variables of the Rössler system (8), the observability indices averaged over a trajectory are

$$\delta_x = 0.022\ (0.022),$$

$$\delta_y = 0.133\ (0.133),$$

$$\delta_z = 0.0063\ (0.011), \tag{13}$$

where the values within parentheses were obtained using the definition used in our previous works [3,5]. From the previous values, the variables can be ranked in descending degree of observability according to $y \rhd x \rhd z$. This order is thus in agreement with the presence of singularities in the embedding coordinate transformations Φ_s. Such singularities lay at the root of the lack of observability. It is interesting to notice that with the definition of observability used in this paper, the practical difficulty of using variable z as an observable has been further emphasized, whereas no change has been verified in the observability indices associated to the two other variables.

A. Observability and global modeling

When a differential embedding is used, it is possible to rewrite the original equations in terms of the derivatives of the observable in the canonical form:

$$\begin{aligned} \dot{X}_1 &= X_2, \\ \dot{X}_2 &= X_3, \\ &\vdots \\ \dot{X}_d &= F_s(X_1, X_2, \ldots, X_d), \end{aligned} \tag{14}$$

where $X_i=s^{(i-1)}$ in the case of a d-dimensional system. The model function F_s can be obtained using the Lie derivatives $\mathcal{L}_f^d h(\boldsymbol{x})$ and the inverse of map Φ. Obviously, the differential model (14) is not defined for the singular sets of map Φ : these singularities occur in the model function F_s as poles of some rational function. The existence of such singularities may prevent the analytically derivation of the model function F_s [6].

When working with differential embeddings, it is thus possible to get a set of differential equations from the measurement of the time evolution of a single observable by estimating the model function F_s as an expansion on a polynomial basis. The estimated model function

$$\tilde{F}_s = \sum_{i=1}^{N_K} K_i \Xi^i, \tag{15}$$

where Ξ^i are polynomial terms of the form $X_1^{k_1} X_2^{k_2} \ldots X_d^{k_d}$. Coefficients K_i are numerically estimated using a least-squares-like technique detailed in [6] whereas the choice of the polynomial terms is still a challenge [21]. Since the estimated function $\tilde{F}_s$ has a polynomial form, it becomes obvious that the efficiency of the approximation directly depends on the number and multiplicity of the poles, just as the observability indices are effected by such singular sets. Other less general representations may be used in the approximation [22] at a higher computational cost. Even if more general representations are used, still the observability will pose real problems in the vicinities of the singular sets. Therefore, the accuracy of the approximation will be strongly influenced by how the neighborhood of a singular set is visited by the trajectory rather than the existence of the pole itself or even the basis functions used to compose the model. This establishes a link between the observability indices and our ability in getting a successful global model from a given observable.

IV. MORE ADVANCED EXAMPLES

In this section, two different systems are investigated. For the sake of comparison, the values computed using the linear observability will be shown in parentheses. The focus of this section is to point out the advantages of using the observability indices as defined in Sec. II A [23,24].

A. The hyperchaotic Rössler system

In 1979, Rössler proposed a four-dimensional system with an hyperchaotic attractor [25]. The equations are

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay + w, \\ \dot{z} &= b + xz, \\ \dot{w} &= -cz + dw, \end{aligned} \tag{16}$$

with parameter values $(a,b,c,d)=(0.25,3,0.5,0.05)$. Initial conditions are $(x_0,y_0,z_0,w_0)=(-10,-6,0,10.1)$. Differential embeddings of the hyperchaotic attractors from the four different variables are shown in Fig. 1. The observability indices are

$$\begin{aligned} \delta_x &= 2.2 \times 10^{-4}(5.6 \times 10^{-4}), \\ \delta_y &= 9.0 \times 10^{-4}(9.4 \times 10^{-4}), \\ \delta_z &= 1.3 \times 10^{-7}(5.3 \times 10^{-4}), \\ \delta_w &= 2.1 \times 10^{-4}(1.3 \times 10^{-3}) \end{aligned} \tag{17}$$

and lead to the order $y \rhd x \rhd w \rhd z$.

This is the first reported example where the observability indices computed using (6) and (7) with (4) rather than approximating matrix A by the system Jacobian matrix. When the latter matrix is used as a local approximation, the observability order is $w \rhd y \rhd x \rhd z$. Such a ranking is difficult to harmonize with the differential embeddings shown in Fig. 1. The differential embedding induced by the z variable is clearly similar to the differential embedding of the z variable of the 3D Rössler system which is already known to be a very poor observable for the dynamics. This fact is correctly quantified by the observability indices using either procedure. On the other hand, the differential embedding induced by the w variable provides a representation of the dynamics where part of the attractor has been squeezed to such an extent that it is quite difficult to distinguish two different revolutions on the attractor. This poor observability from the w variable is better quantified by the procedure used in this paper when compared to the results obtained with the former definition.

In terms of the map between the original phase space $\mathbb{R}^4(x,y,z,w)$ and the differential embedding $\mathbb{R}^4(X_1,X_2,X_3,X_4)$ induced by $s(t)=h(x,y,z,w)$, the singularities involved in the model function are in good agreement with these indices computed using the new definition. These singularities correspond to the set where $\det(\Phi_s)$ vanishes and therefore no local inverse is defined. This, as discussed at the end of Sec. II A and in Sec. III A, characterizes regions in phase space over which the system is practically unobservable.

The singularities of the model functions obtained by successively embedding the hyperchaotic Rössler system from each state variable are reported in Table I. From this table, the two extreme situations are clearly recognized: the model function F_y has no singularities when the embedding is attempted using $y(t)$ as the observable and an order-3 singularity appears in F_z when the embedding is attempted using $z(t)$ as the observable. On the other hand, it is quite difficult to distinguish the type of singularities involved in the model functions F_x and F_w, a fact that is confirmed by the relative similarity of the observability indices δ_x and δ_w. It is noteworthy that such a similarity is not observed if the former definition for computing the observability indices is used.

B. The Hénon-Heiles system

The Hénon-Heiles system [26], a two-degrees-of-freedom conservative system, is governed by the set of four ordinary differential equations

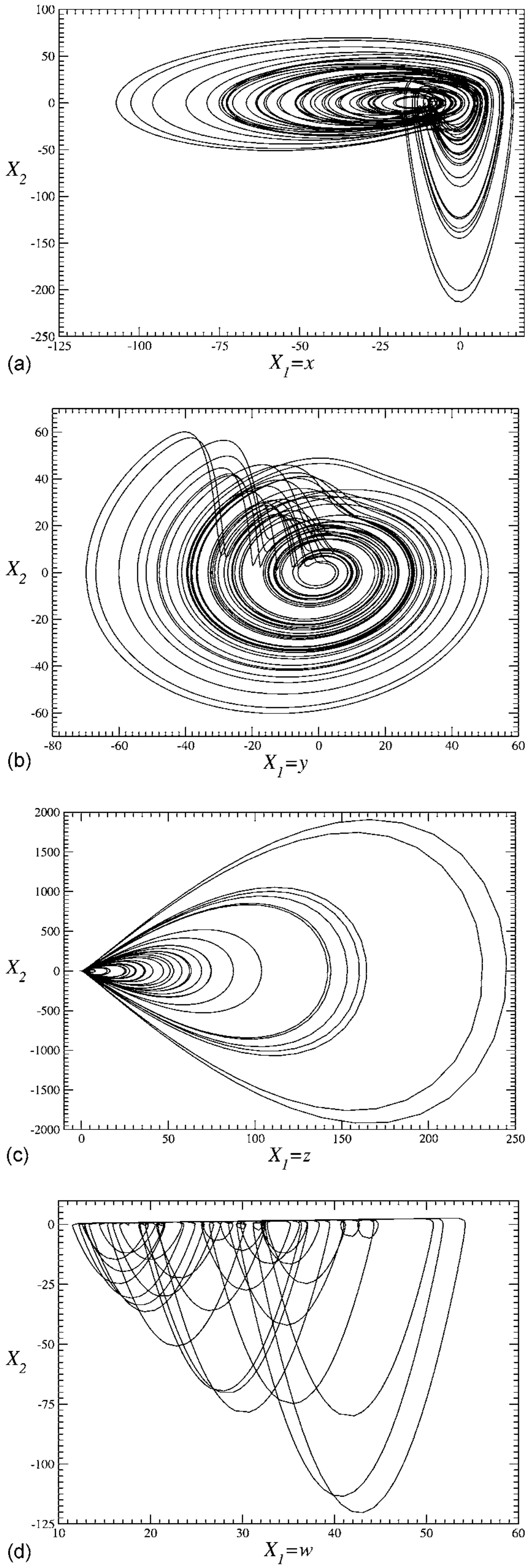

FIG. 1. Plane projections of the four differential embeddings induced by the variable of the hyperchaotic Rössler system. Parameter values: $(a,b,c,d)=(0.25,3,0.5,0.05)$.

TABLE I. Singularities involved in the model function of the canonical forms of the hyperchaotic Rössler system induced by the different variables.

Variable	Denominator of function F_s
y	constant
x	$X^2-(a+d)X+(ad-c)+Y$
w	$c(dX-Y)^2$
z	X^3

$$\dot{x}=u,$$

$$\dot{y}=v,$$

$$\dot{u}=-x-2xy,$$

$$\dot{v}=-y+y^2-x^2. \tag{18}$$

In this work the energy E of the system is equal to 0.128546999 for the initial conditions $(x_0,y_0,u_0,v_0)=(0.0,0.67,0.093,0.0)$. The observability indices for this case are

$$\delta_x=0.041(0.087),$$

$$\delta_y=0.026(0.059),$$

$$\delta_u=0.022(0.006),$$

$$\delta_v=0.015(0.007). \tag{19}$$

The first remark is that using (6) and (7) with (4), the variables can be ranked as their corresponding momenta, that is, $x \triangleright y$ and $u \triangleright v$. This comes as no surprise since $\dot{x}=u$ and $\dot{y}=v$. Note that this was not the case when the jth power of the Jacobian matrix was used instead of the A^j matrices.

From the differential embeddings of this system, it appears that the differential embedding induced by the x variable is the simplest one in terms of the folding mechanism [Fig. 2(a)]. The y-induced embedding reveals a region (the extreme left of the portrait) where different revolutions in the phase space are not distinguishable [Fig. 2(b)]. Such a feature is always the root of a decrease in observability. The two embeddings induced by the momenta are more complicated, that is, they have a more folded structure [Figs. 2(c) and 2(d)], a feature associated with an overall decrease of observability.

When maps Φ_s between the original phase space $\mathbb{R}^4(x,y,u,w)$ and the differential embeddings $\mathbb{R}^4(s,\dot{s},\ddot{s},\dddot{s})$ are considered, again there is a strong connection between the observability indices and the singularities of the inverse of such maps. When inverted, the map

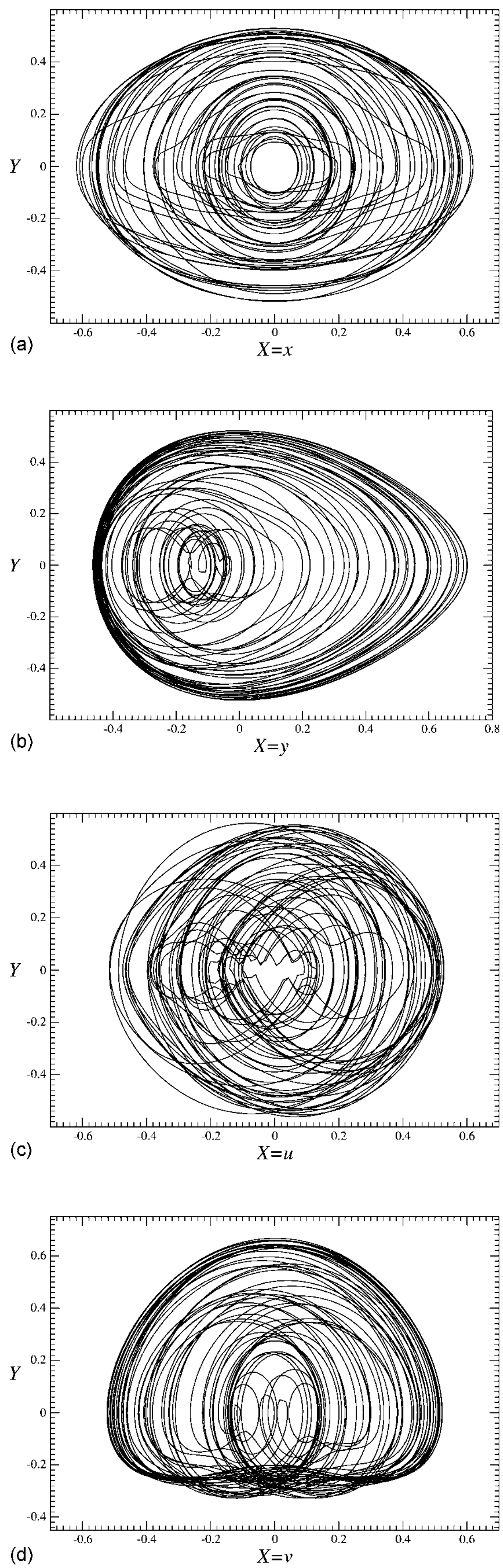

FIG. 2. Differential embeddings induced by the variables of the Hénon-Heiles system.

$$\Phi_x^{-1} = \left| \begin{array}{l} x = X, \\ y = -\dfrac{X+Z}{2X}, \\ u = Y, \\ v = -\dfrac{YZ+XW}{2X^2} \end{array} \right. \tag{20}$$

associated with the x-induced embedding has an order-2 singularity in $X^2=0$. The model function has this singularity ($X^2=0$). This, on its own right, is already a great difficulty when it comes to global model building. Added to this difficulty is the fact that a model for this system should also be conservative, a constraint which is difficult to impose in practice.

When the y variable is the observable, the situation becomes slightly more intricate since the map

$$\Phi_y^{-1} = \left| \begin{array}{l} x = \pm\sqrt{X(X-1)-Z}, \\ y = X, \\ u = \mp\dfrac{Y(2X-1)-W}{2\sqrt{X(X-1)-Z}}, \\ v = Y \end{array} \right. \tag{21}$$

has a more complicated singularity. The model function to estimate has a singularity equal to (X^2-X-Z), that is, an order-2 singularity. The intersection set between the singular set and the differential embedding is clearly larger for the y induced differential embedding than for the x-induced differential embedding (Fig. 3). This means that the system spends more time in the vicinity of an unobservable region when observed from the y variable than when it is observed from the x variable.

When the two momenta, u and v, are considered, the structure of the phase portrait becomes more folded [Figs. 2(c) and 2(d)]. This results from the derivation. The structure of maps Φ_u^{-1} and Φ_v^{-1} involve higher order singularities. Consequently, the observability indices from the momenta are slightly less than for their corresponding variables. The momenta are ranked as their corresponding variables. In this conservative system, the relationships between the observability indices and the complexity of the couplings between the dynamical variables has, therefore, been verified.

V. CONCLUSION

This paper has discussed issues relating to the observability of nonlinear dynamical systems. The results presented extend previous definitions of observability indices using Lie derivatives. With this definition it was possible to establish an apparently missing link between observability and an embedding theory. Indeed, it has been shown that the observability matrix is, in fact, the Jacobian matrix of the coordinate transformation between the original phase space and the differential embedding induced by the particular observable. A consequence of this is that the loss of observability can be

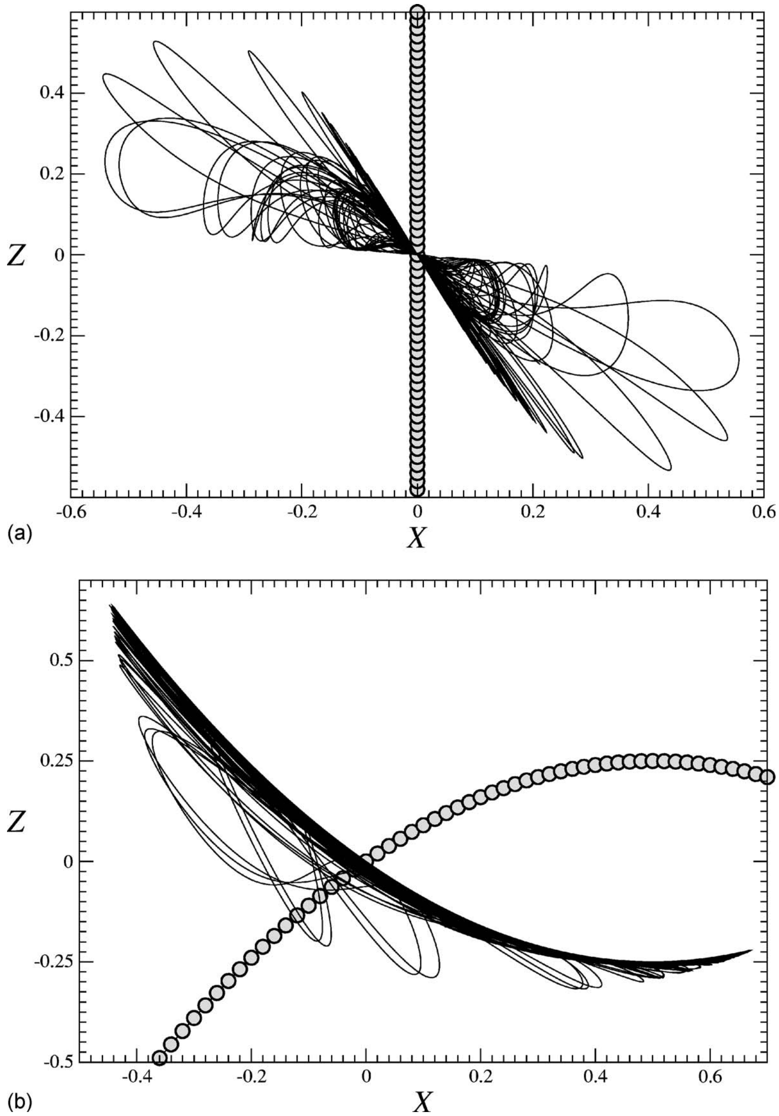

FIG. 3. $X-Z$ plane projection of the x and the y-induced differential embeddings of the Hénon-Heiles system. The circles indicate the singular set. Clearly, when observed from the y variable, the intersection of the trajectory and the singular set is larger [plot (b)] than for the case when the system is observed from the x variable [plot (a)].

associated to the occurrences of singularities in the inverse embedding maps.

Two new examples have been discussed and it has been argued that the definition of the observability matrix used in this paper, in many cases, provides better indications concerning observability. A practical consequence of this is that observability has a direct bearing on the ability to obtain a global model from a given observable and to choose the driving signal in Kalman-based state estimation problems.

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